Dr. T. Centeleghe, Dr. J. M. Cerviño, Dr. N. Kumar[1] 16. April 2012

Working group basics-seminar SS2012 :

Automorphic forms and representations of GL_2

Wednesdays from 9:15 to 10:45, INF 368, room 248[2]

The goal of this seminar is to study the basics of automorphic forms and representations of GL_2 following Bump's book [B], Chapters 4 and 3 (with some omissions). The character of the seminar is rather elementary – at least for the experts.

Please: As you can see, we have a total of 15 lectures, as many as seminar meetings in this summer semester 2012. In particular, there is no room for extra time at all! We, the organizers, have done our best to split up the talks in such a way, that every single one fits in one lecture. So, we kindly ask you to try to stay within the schedule.

Any citation or notation without a particular reference is from [B].

---

1. Representations of GL_2 over a finite field

The aim of this lecture is to introduce Mackey theory, the notion of cuspidal representations, and to prove the uniqueness of Whittaker models.

- Introduce the required notions before Proposition 4.1.2 and prove this proposition.
- State and Prove Lemma 4.1.1, Theorem 4.1.1, and briefly mention Proposition 4.1.4.
- The notion of cuspidal representations, briefly prove Proposition 4.1.5, Proposition 4.1.6.
- Introduce Whittaker models and Prove Theorem 4.1.2.

Literature: [B, §4.1]

Speaker: Yamidt Bermudez-Torbón 18.4.

---

2. Smooth and Admissible representations

The goal of this lecture is to prove Theorem 4.2.2 along with other important notions.

- Define smooth, admissible representations, and establish their basic properties. Prove Proposition 4.2.2.
- Define Hecke algebras and their representations, prove Proposition 4.2.3, Proposition 4.2.5.
- Discuss the material from page 431, state and prove Theorem 4.2.1 (assuming Proposition 4.2.7).
- Prove Theorem 4.2.2, assuming Theorem 4.2.3. As corollaries, sketch the proofs of Proposition 4.2.8 and Proposition 4.2.9.

Literature: [BH], [B, §4.2]

Speaker: Yujia Qiu 25.4.

---

[1] The role of the organizers is merely a supportive one whilst the seminar, since the road was already traced by D. Bump in his book.

[2] Begin: April 18.

3. Distributions and Sheaves

We give an overview of §4.3 and prove Theorem 4.2.3 in the case $n = 2$. We need these to extend Mackey theory.

- Introduce distributions, state Proposition 4.3.1, and define actions of G on distributions.
- Introduce definitions (e.g., Invariant measure) to state Proposition 4.3.5.
- Introduce notions (e.g., Sheaves, Sheaf of distributions) to state Proposition 4.3.15.
- Give a proof of Theorem 4.2.3 for $n = 2$.

Literature: [B, §4.3].

Speaker: Andreas Maurischat 2.5.

---

4. Whittaker models and Jacquet functor

The aim of this lecture is to give a proof of the uniqueness of Whittaker models (from now on in the *local* case), a fundamental result which will be applied in a later lecture for multiplicity one for automorphic forms and the construction of L-functions.

- Define Whittaker models, give a sketch of proof of Theorem 4.4.2 and prove Theorem 4.4.1.
- Introduce the Jacquet functor and the twisted Jacquet functor, sketch of proof of Proposition 4.4.1, and state the properties of the (twisted) Jacquet functor.
- State and prove Theorem 4.4.3, Proposition 4.4.7 and introduce Kirillov model.
- Introduce the notions before Lemma 4.4.2, Theorem 4.4.4 and state the lemma and theorem, if possible prove them.

Literature: [B, §4.4].

Speaker: NN 9.5.

---

5. and 6. Principal series representations

These two talks form a block aiming to present the most relevant results from [B, §4.5].
The first of these two talks aims at defining and studying the most basic properties of *principal series representations*. The first main focus should be on Proposition 4.5.4 (the representations $\mathcal{B}(\chi_1, \chi_2)$ admit at most one Whittaker model). The second main goal is the irreducibility criterion, Theorem 4.5.1.

- Define induced representations, and prove everything until Proposition 4.5.5 – with the exception of the proof of the Iwasawa decomposition.
- Sketch Proposition 4.5.5 and go on with all proofs until the last paragraph of page 477.

The second talk has two main goals. The first one is Theorem 4.5.3 which extends the result of the previous Theorem 4.5.2 from last talk (on intertwining operators for irreducible principal series representations). The second is the determination of the Jacquet modules for these type of representations.

- Prove all the statements from the last paragraph of page 477 till the end of the proof on page 484.

Both the first and second talk of this block have as *extra bonus* the opportunity to solve Exercises 4.5.2 (first talk), 4.5.3 and 4.5.4 (last two for the second talk) – which should be presented!

Literature: [B, §4.5], cf. also [BH, Chapter 3].

Speaker(s): NN 16.5. and 23.5.

---

7. Spherical representations

We study *spherical representations* – a particularly important class, since global representations give rise to these at almost all places!
The main results to be presented here are Theorems 4.6.3, 4.6.4 and 4.6.7.

- Go through all the proofs until Proposition 4.6.4.
- Simply write down Equation (6.3), in few words explain the Proposition 4.6.5 as a direct consequence of this equation – *without any further comment*, since there is still a lot to be done!
- Proceed with statements and proofs until the last paragraph of page 497.
- After making a remark on Iwahori-fixed vectors (cf. page 501), resume on page 507 with all proofs till the end of the section.

Literature: [B, §4.6].

Speaker: Patrik Hubschmid 30.5.

---

8. *Intermezzo:* Tate's thesis

The aim of this lecture is to make the audience to get acquainted with Tate's thesis and its importance.

- Briefly introduce the adelic language, e.g., adeles, ideles, and few properties of them.
- Recall Fourier analysis on locally compact groups, Poisson summation formula.
- Define local Zeta functions and define L-functions in the archimedean theory.
- Finally, define global Zeta functions, state its properties, and show how Tate's thesis unified the classical results.

Literature: [T] (Main reference), [G] and [P] (for an overview), and [B, §3.1], [K2], [RV] (additional references).

Speaker: Tommaso Centeleghe 6.6.

---

9. Local functional equations

The main goal for this talk is to prove Theorem 4.7.5.
Start by proving Proposition 4.7.3 and use/sketch the two previous ones to prove Theorem 4.7.1. Prove Proposition 4.7.4, Theorem 4.7.2 (may show only one case: the second?). Follow further the exposition as in the book, and if necessary leave also some details (as Bump does) for the audience; then include the proof of Proposition 4.7.6, after which one may stop.

- Propositions 4.7.1 (proof?), 4.7.3 (with proof!), Theorem 4.7.1.
- Proposition 4.7.4, Theorem 4.7.2 (at least one case).
- Complete the statement of the previous theorem by mentioning Theorem 4.7.3, Theorem 4.7.4.
- Prove Proposition 4.7.5 and Theorem 4.7.5.
- Hopefully we see the proof of Proposition 4.7.6.

Literature: [B, §4.7].

Speaker: Chandrakant Aribam 13.6.

---

10. AND 11. SUPERCUSPIDAL REPRESENTATIONS – LOCAL LANGLANDS CORRESPONDENCE

In this two talks we study the "last" possible type of irreducible, admissible representations: the *supercuspidal representations*. We construct the them as induced representations from open subgroups (being a theorem of Kutzko that all supercuspidals arise in this way). In the second talk we use the Weil representation to construct a particular type of supercuspidal representations, so called *dihedral*[3]. We add half an hour to the second talk to state the local Langlands correspondence.

- Begin with Proposition 4.8.1, Theorem 4.8.1.
- State Lemma 4.8.1 (maybe without proof) and prove the theorem of Segal-Shale-Weil (Theorem 4.8.2) – might be a good point for finishing the first talk.
- Prove Theorem 4.8.3, sketch Theorem 4.8.5 and finish with the proof of Theorem 4.8.6. Be short in the explanations before Theorem 4.8.3, and in the following until last theorem (particularly with Weil's integral interpretation of the Hilbert symbol) – one may also skip mentioning the Howe correspondence, which we have studied up to some detail last semester.
- Introduce representations of the Weil-Deligne group and state the Local Langlands correspondence (for GL_2 as in [D, §3, p. 92–96], and if possible complete the statement for GL_n from [K1, p. 380]).

Exercises 4.5.5–4.5.7 are needed.

Literature: [B, §4.8 and §4.5], [D, §3, p. 92–96], [K1, p. 380].

Speaker(s): NN. and Alexander Ivanov 20.6. and 27.6. (90+30 min.)

---

12. CLASSICAL AUTOMORPHIC FORMS AND REPRESENTATIONS

This lecture has two goals: the first one is to introduce the spaces of automorphic forms $\mathcal{A}(\Gamma\backslash G, \chi, \omega)$ and $\mathcal{A}_0(\Gamma\backslash G, \chi, \omega)$, and to explain their relationship with the classical notions of holomorphic modular forms and of Maass forms. The second goal is to sketch the proof of Theorem 3.2.2 and of Theorem 3.2.3 on the discrete part of the spectrum of G. The main reference is §3.2, however it is necessary at several places to link the material of the section with the representation theory explained in §2.2, 2.3. The second part of the lecture is the analogue of §2.3 in the case when $\Gamma\backslash\mathcal{H}$ is non-compact. (Warning: the book has several typos when referring to previous theorems or propositions).

- Discuss the right representation of G on $C^\infty(G)$, the corresponding derived action of $\mathfrak{g}$ (cf. eq. (2.1)), and that of $U(\mathfrak{g})$. Describe the center $\mathcal{Z}$ of $U(\mathfrak{g})$ (cf. §2.2 for all the missing details).
- Introduce the notions of K-finite and of $\mathcal{Z}$-finite vector for an element $F \in C^\infty(G)$, and the notion of $(\mathfrak{g}, K)$-module. Define the spaces $\mathcal{A}(\Gamma\backslash G, \chi, \omega)$ and $\mathcal{A}_0(\Gamma\backslash G, \chi, \omega)$ and discuss Theorem 3.2.1 (explain how the theorem follows from Theorem 2.9.2 from §2.9).
- Discuss the relationship between the automorphic forms just introduced and the classical notions of holomorphic modular forms and Maass forms.
- Sketch the proof of Theorem 3.2.2 and of Theorem 3.2.3. For the first one, one needs to go over Theorem 2.3.3 from §2.3 and make use of Proposition 3.2.3, which should be discussed as well.

---

[3] It is a theorem of Tunnel, that in the GL_2 case all supercuspidal representations are dihedral – cf. page 549 for the references.

Literature: [B, §3.2, §2.2, §2.3].

Speaker(s): Ann-Kristin Juschka 4.7.

---

13. Automorphic Representations and Adelization

In this lecture the adelic $GL(n)$ is introduced, and the main theorems of the previous section are adapted to this setting (proofs are only given for $n = 2$ and $F = \mathbb{Q}$). The notions of automorphic representation and admissible representations are given, and it is explained how the discrete spectrum of $GL(n, \mathbb{A})$ gives rise to admissible representations (Thm. 3.3.4). The lecture ends with the description of the adelization of classical automorphic forms.

- Describe the adelic $\mathrm{GL}(n)$ over a number field, state Theorem 3.3.1, and prove Proposition 3.3.1 and Proposition 3.3.2.
- Introduce $L^2(\mathrm{GL}(n, F)\backslash \mathrm{GL}(n, \mathbb{A}), \omega)$ and $L_0^2(\mathrm{GL}(n, F)\backslash \mathrm{GL}(n, \mathbb{A}), \omega)$, prove Proposition 3.3.3 and deduce Theorem 3.3.2 (they are the analogue of Prop. 3.2.3 and Thm. 3.2.2 from the previous section, proofs are similar).
- Introduce the spaces $\mathcal{A}(\mathrm{GL}(n, F)\backslash \mathrm{GL}(n, \mathbb{A}), \omega)$ and $\mathcal{A}_0(\mathrm{GL}(n, F)\backslash \mathrm{GL}(n, \mathbb{A}), \omega)$. Explain the concepts of automorphic representation, admissible representation of $\mathrm{GL}(n, \mathbb{A})$, and of restricted tensor product.
- State Theorem 3.3.3, prove Theorem 3.3.4 and state Theorem 3.3.6.
- Describe the adelization of classical automorphic forms by going over §3.6.

Literature: [B, §3.3, §3.6].

Speaker(s): Narasimha Kumar 11.7.

---

14. The Tensor Product Theorem

In this lecture the formalism of *idempotented* algebras is developed, the construction of the local Hecke algebra $\mathcal{H}_G$ for $G = \mathrm{GL}(2)$ over a local field is discussed, and the theorem on the structure of simple admissible modules over idempotented algebras is proved (Theorem 3.4.4). The tensor product theorem then follows as a special case. Section 3.4 of the book should be followed quite closely. Several auxiliary results will only be stated.

- Define idempotented algebras $(H, \mathcal{E})$ over Ω, give the notion of smooth and admissible module. Briefly recall the construction of the Hecke algebra $\mathcal{H}_G$ for a TDLC group G (cf. §4.2).
- Construct $\mathcal{H}_K$ and $\mathcal{H}_G$ for a compact Lie group K and a reductive Lie group G. Include Proposition 3.4.3 and discuss without proof Proposition 3.4.4.
- State Theorem 3.4.2, prove Proposition 3.4.8 and deduce Proposition 3.4.9.
- Prove Theorem 3.4.3 and 3.4.4, deduce Theorem 3.3.3.

Literature: [B, §3.4].

Speaker(s): Ralf Butenuth 18.7.

---

15. Whittaker Models and Automorphic Forms

In this lecture existence and uniqueness of global Whittaker models for irreducible admissible representations of $\mathrm{GL}(2, \mathbb{A})$ are shown. A preliminary step is to give an equivalent formulation of the local multiplicity one result for $\mathrm{GL}(2)$ in the non-archimedean setting (Thm. 3.5.1) which is suitable for the archimedean as well (Prop. 3.5.1).

- State Theorem 3.5.3 and show its equivalence to Theorem 3.5.1 (this theorem was proved in Lecture 4).
- State Proposition 3.5.1, point out its equivalence with Theorem 3.5.3 in the non-archimedean case.
- Prove Proposition 3.5.2 (for the archimedean case one will have to refer to results from Chapter 2).
- Prove Theorem 3.5.4 and Theorem 3.5.5, mention how one deduces Theorem 3.3.6 from these.

Literature: [B, §3.5].

Speaker(s): David Guiraud 25.7.

REFERENCES

[B] D. Bump, *Automorphic forms and representations*, Cambridge Studies in Advanced Mathematics, 55, Cambridge Univ. Press, Cambridge, 1997.

[BH] C. J. Bushnell and G. Henniart, *The local Langlands conjecture for* $\mathrm{GL}(2)$, Grundlehren der Mathematischen Wissenschaften, 335, Springer, Berlin, 2006.

[D] P. Deligne, Formes modulaires et représentations de $\mathrm{GL}(2)$. (French) *Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, pp. 55–105. Lecture Notes in Math., Vol. 349, Springer, Berlin, 1973.

[G] S. Gelbart, *Langlands picture of Automorphic forms and L-functions*, Lecture notes at Shadong University, 2009.

[K1] S. S. Kudla, The local Langlands correspondence: the non-Archimedean case. *Motives* (Seattle, WA, 1991), 365–391, Proc. Sympos. Pure Math., 55, Part 2, Amer. Math. Soc., Providence, RI, 1994.

[K2] S. S. Kudla, Tate's thesis, in *An introduction to the Langlands program (Jerusalem, 2001)*, 109–131, Birkhäuser, Boston, 2003.

[P] D. Prasad, "Lectures on Tate's Thesis", ICTP Lecture Notes, 2007.

[RV] D. Ramakrishnan and R. J. Valenza, *Fourier analysis on number fields*, Graduate Texts in Mathematics, 186, Springer, New York, 1999.

[T] J. T. Tate, Fourier analysis in number fields, and Hecke's zeta-functions, in *Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965)*, 305–347, Thompson, Washington, DC, 1967.