GENERALIZED THETA SERIES AND SPHERICAL DESIGNS

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To the memory of Boris Venkov

Abstract. We investigate the relation between the vanishing order at infinity of certain cusp forms associated to integral lattices and shells of lattices being spherical t-designs. An explicit criterion is proved and illustrated on root lattices of type A, D and E, on the Leech lattice and on some other extremal modular lattices.

1. Introduction

Let $\Lambda \subset \mathbb{E}^n$ be an integral lattice. The l-shell $\Lambda_{(l)}$ of the lattice consists by definition of all lattice vectors $\lambda \in \Lambda$ such that $\|\lambda\|^2 = l$. If the vectors of the l-shell are "equally distributed" we can use them to calculate approximations for the integral of functions on the sphere as follows:

$$\int_{S^{n-1}} f d\mu \approx \frac{1}{\#(\Lambda_{(l)})} \sum_{\lambda \in \Lambda_{(l)}} f\left(\frac{\lambda}{\|\lambda\|}\right) \tag{1.1}$$

where $d\mu$ stands for the $\mathrm{O}(n)$-invariant measure on the sphere normalized such that $\int_{S^{n-1}} d\mu = 1$. The l-shell $\Lambda_{(l)}$ of Λ is defined to be a *t-design* when (1.1) becomes an equality for all polynomials $f : \mathbb{E}^n \to \mathbb{C}$ of total degree at most t.

To give an example: the 1-shell of the lattice $\mathbb{Z}^2 \subset \mathbb{E}^2$ is a spherical 3-design. Indeed, we must check that for all $f \in \{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3\}$ — this is a basis for the vector space of polynomials of degree at most three in two variables — we have

$$\int_{S^1} f d\mu = \frac{f(1,0) + f(0,1) + f(-1,0) + f(0,-1)}{4}.$$

However, this 1-shell is not a spherical 4-design, because for the function $f(x,y) = x^4$ one has

$$\int_{S^1} f d\mu = \frac{3}{8} \neq \frac{1}{2} = \frac{f(1,0) + f(0,1) + f(-1,0) + f(0,-1)}{4}.$$

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This is a very naive check which works without much effort for this particularly small example.

More sophisticated criteria for testing whether l-shells, or more generally finite discrete subsets of the sphere, are spherical t-designs were studied for example in [7] and in [1]. In the latter, the theory of theta series with harmonic coefficients and its connection with Jacobi forms is used to classify some extremal lattices.

With the introduction of lattice invariants from [4] we have at hand a more systematic way of studying properties of lattices. In particular, we show that for an integral lattice Λ, the invariants $\Theta_{m,m;\Lambda}$ for $m \geq 0$ an integer[1] , capture the information of an l-shell being a spherical t-design.

The paper is organized as follows. In Section 2 we introduce the *generalized theta series*[2] $\Theta_{m,m}$ and give some of its properties. The main result is Theorem 3.3 which formulates the property of being a spherical t-design in terms of Fourier coefficients of the modular forms $\Theta_{m,m}$. Since the generalized theta series $\Theta_{m,m}$ can be expressed using the angles between pairs of lattice vectors, it turns out that being a spherical t-design can be expressed in such terms; cf. [7, Théorème 3.2]. We apply our criteria to root lattices of type ADE in Section 4, to the Leech lattice and some other extremal modular lattices in the last section.

2. The generalized theta series $\Theta_{m,m}$

2.1. **Harmonic polynomials on $\mathbb{E}^n$.** We recall some basic facts about harmonic polynomials – see for example [5, Chapter XIII, Exercises 33–35]. Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a polynomial on $\mathbb{E}^n$. The left action of the orthogonal group $\mathrm{O}(n)$ on $\mathbb{E}^n$ induces a right action on $\mathbb{R}[x_1, \ldots, x_n]$ where a pair $(f, \gamma) \in \mathbb{R}[x_1, \ldots, x_n] \times \mathrm{O}(n)$ is mapped to f^γ with $f^\gamma(x) = f(\gamma(x))$. This action respects the degree of polynomials, so we can assume that f is homogeneous. We write $\mathrm{Pol}(n, d)$ for the real vector space of homogeneous polynomials in $\mathbb{R}[x_1, \ldots, x_n]$ of degree d. On $\mathrm{Pol}(n, d)$ we have a bilinear form given by

$$\langle f, g \rangle = f\left(\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \ldots, \frac{\partial}{\partial X_n}\right) g(X_1, X_2, \ldots, X_n)\,.$$

We check that the monomials form an orthogonal basis with respect to this positive definite form. Furthermore, the form is $\mathrm{O}(n)$-invariant, i.e. $\langle f^\gamma, g^\gamma \rangle = \langle f, g \rangle$. To obtain the irreducible components of the representation of $\mathrm{O}(n)$ on $\mathrm{Pol}(n, d)$ we need the harmonic polynomials. We call f *harmonic* if it satisfies the differential equation $\Delta(f) = 0$ with $\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ the Laplace operator. The kernel of the surjection $\Delta : \mathrm{Pol}(n, d) \to \mathrm{Pol}(n, d-2)$ is the vector space

[1] Professor R. Schulze-Pillot kindly informed us that the invariant $\Theta_{1,1}$, at least in dimension 4, already appeared in [2] – cf. (5.3) loc.cit..

[2] This term is usually used in the literature for any theta series with harmonic coefficients – we prefer to use this term only for those (sums of products of) theta series with harmonic coefficients which are invariant under the orthogonal group.

$\mathrm{Harm}(n,d)$ of homogeneous harmonic polynomials of degree d. Denoting by $r^2 = r^2(X_1,\dots,X_n) = \sum_{i=1}^n X_i^2$ the generator of $\mathbb{R}[x_1,\dots,x_n]^{\mathrm{O}(n)}$ we obtain the decomposition of $\mathrm{Pol}(n,d)$ into irreducible representations as follows:

$$\mathrm{Pol}(n,d) = \bigoplus_{k=0}^{\lfloor \frac{d}{2} \rfloor} r^{2k}\mathrm{Harm}(n,d-2k)\,. \tag{2.1}$$

For a harmonic function $h \in \mathrm{Harm}(n,d)$ and an integral lattice $\Lambda \subset \mathbb{E}^n$ we define the harmonic theta series $\Theta_{h;\Lambda}$ as a complex valued function on the upper half plane as

$$\Theta_{h;\Lambda}(z) := \sum_{\lambda\in\Lambda} h(\lambda)q^{\|\lambda\|^2},\; q = q(z) = \exp(2\pi i z).$$

If the degree d of h is odd, then we deduce from $h(-\lambda) = (-1)^d h(\lambda)$ that $\Theta_{h;\Lambda} \equiv 0$. So we assume hereafter that $d = 2m$ is an even integer.

2.2. **Invariant tensors.** Next we introduce the following bihomogeneous polynomials on $\mathbb{E}^n \times \mathbb{E}^n$ given by

$$p_m(x,y) = \sum_{k=0}^{m} \frac{(-1)^k \|x\|^{2k}\|y\|^{2k}\langle x,y\rangle^{2m-2k}}{(2m-2k)!k!2^k \prod_{l=0}^{k-1}(n+4m-4-2l)}$$

for all integers $m \geq 0$. It is obvious from the definition that p_m is invariant under the orthogonal group. Hence, $p_m \in \left(\mathrm{Pol}(n,2m)\otimes \mathrm{Pol}(n,2m)\right)^{\mathrm{O}(n)}$. The polynomial p_m is the unique bihomogeneous polynomial of bidegree $(2m,2m)$, unique up to a scalar, which is $\mathrm{O}(n)$-invariant and satisfies the two differential equations

$$\Delta_x p_m = 0, \quad \text{and} \quad \Delta_y p_m = 0.$$

So we have $p_m \in \left(\mathrm{Harm}(n,2m)\otimes \mathrm{Harm}(n,2m)\right)^{\mathrm{O}(n)}$.

2.3. **The modular forms $\Theta_{m,m}$.** We will need Theorem 3.3 from our article [4]. Using the polynomial p_m just defined, we restate the mentioned theorem as follows.

Theorem 2.1. *Let $\Lambda \subset \mathbb{E}^n$ be integral lattice of level N and $m > 0$ an integer. The function*

$$\Theta_{m,m}(\Lambda) := \Theta_{m,m;\Lambda} := \sum_{h\in\mathcal{B}_{\mathrm{harm}}} \Theta_{h;\Lambda}^2,$$

where $\mathcal{B}_{\mathrm{harm}}$ is a orthonormal basis of $\mathrm{Harm}(n,2m)$*, is a modular form of weight $n+4m$ and level N. The q-expansion of $\Theta_{m,m;\Lambda}$ is given by*

$$\Theta_{m,m;\Lambda}(z) = \sum_{(x,y)\in\Lambda\times\Lambda} p_m(x,y)q^{\|x\|^2+\|y\|^2}.$$

3. Characterizations of t-designs

Suppose $\Lambda \subset \mathbb{E}^n$ is an integral lattice. We assume that the l-shell of Λ is not empty – see Remark (1) following next lemma. We say that the l-shell $\Lambda_{(l)}$ of Λ is *spherical for* $\mathrm{Harm}(n,d)$, when for all homogeneous harmonic polynomials f of degree d we have $\sum_{\lambda\in\Lambda_{(l)}} f(\lambda) = 0$.

The following well known observation is often used in the literature to give an alternative definition of spherical t-designs – which works actually for any finite subset on the sphere, not necessarily a lattice shell. We include a proof here for the sake of completeness.

Lemma 3.1. *The l-shell of a lattice $\Lambda \subset \mathbb{E}^n$ is a spherical t-design if and only if $\Lambda_{(l)}$ is spherical for $\mathrm{Harm}(n,d)$ for all integers d such that $1 \le d \le t$ holds.*

Proof. Assume that $\Lambda_{(l)}$ is a spherical t-design. If $f \in \mathrm{Harm}(n,d)$ for some integer d satisfying $1 \le d \le t$, then we have by definition that

$$\int_{S^{n-1}} f d\mu = \frac{1}{\#(\Lambda_{(l)})} \sum_{\lambda\in\Lambda_{(l)}} f(\lambda).$$

By the mean value theorem for harmonic functions, the left hand side equals $f(0)$. Since, f is homogeneous of positive degree we conclude that $f(0) = 0$ which shows that $\Lambda_{(l)}$ is spherical for $\mathrm{Harm}(n,d)$ for all $d = 1,\ldots,t$.

Now assume that $\Lambda_{(l)}$ is spherical for $\mathrm{Harm}(n,d)$ for all $d = 1,\ldots,t$. Let f be a homogeneous polynomials of degree $\deg(f) \le t$. We take the decomposition

$$f = \sum_{k=0}^{\lfloor \deg(f)/2 \rfloor} r^{2k} \cdot f_{\deg(f)-2k}$$

with $f_{\deg(f)-2k} \in \mathrm{Harm}(n, \deg(f)-2k)$ corresponding to the decomposition given in Equation (2.1). Using this equation we deduce from the mean value theorem for harmonic functions that

$$\int_{S^{n-1}} f d\mu = \int_{S^{n-1}} \sum_{k=0}^{\lfloor \deg(f)/2 \rfloor} f_{\deg(f)-2k} d\mu = \sum_{k=0}^{\lfloor \deg(f)/2 \rfloor} f_{\deg(f)-2k}(0) = f_0(0)\,.$$

Since $\Lambda_{(l)}$ is spherical for $\mathrm{Harm}(n,d)$ for all $d = 1,\ldots,t$, we have $\sum_{\lambda\in\Lambda_{(l)}} f_{\deg(f)-2k}$ equal zero for all $k < \frac{\deg(f)}{2}$. The only remaining part is the constant function f_0. This shows the assertion. □

Remark 3.2. (1) After this lemma, one could also regard empty shells $\Lambda_{(l)}$ as a "trivial" spherical design, by simply rewriting Equation (1.1) as

$$\#(\Lambda_{(l)}) \int_{S^{n-1}} f d\mu \approx \sum_{\lambda\in\Lambda_{(l)}} f\left(\frac{\lambda}{\|\lambda\|}\right),$$

which gives always an equality for $\Lambda_{(l)}$ empty.

(2) When f is homogeneous of odd degree d, we have by the above argument that $\int_{S^{n-1}} f d\mu = 0$. On the other hand, since $\lambda \in \Lambda_{(l)}$ implies that $-\lambda$ belongs to the same shell, and in this case $f(\lambda)$ and $f(-\lambda)$ sum up to zero, we have that for all integers $l \geq 0$ the l-shell of any lattice $\Lambda \subset \mathbb{E}^n$ is spherical for $\mathrm{Harm}(n, d)$.

Our main result is the following

Theorem 3.3. *Let $\Lambda \subset \mathbb{E}^n$ be an integral lattice and L be a positive integer. We write the q-expansions of the generalized theta series $\{\Theta_{m,m;\Lambda}\}_{m\geq 1}$ of Λ as*

$$\Theta_{m,m;\Lambda}(z) = \sum_{n\geq 0} a_{m,n} q^n .$$

The following two statements are equivalent.

(1) *The shell $\Lambda_{(l)}$ is a spherical t-design for all $l \leq L$.*

(2) *The Fourier coefficients satisfy $a_{m,n} = 0$ for all pairs (m, n) with $2m \leq t$ and $n \leq 2L$.*

Proof. (1) $\Longrightarrow$ (2). We assume first that the l-shells form spherical t-designs for all $1 \leq l \leq L$. We fix an integer m such that $1 \leq m \leq t/2$ holds. The generalized theta series $\Theta_{m,m;\Lambda}$ is by Theorem 2.1 given as $\Theta_{m,m;\Lambda} = \sum_{h\in\mathcal{B}_{\mathrm{harm}}} \Theta^2_{h;\Lambda}$ with $\mathcal{B}_{\mathrm{harm}}$ an orthonormal basis of $\mathrm{Harm}(n, 2m)$. Since $\Lambda_{(l)}$ is a spherical $2m$-design for all $l = 1, \ldots, L$ we conclude from Lemma 3.1 that $\Theta_{h;\Lambda} \in q^{L+1} \cdot \mathbb{R}[[q]]$ for any harmonic h of degree $2m$. Thus, we have $\Theta^2_{h;\Lambda} \in q^{2L+2} \cdot \mathbb{R}[[q]]$. Therefore $\Theta_{m,m;\Lambda} \in q^{2L+2} \cdot \mathbb{R}[[q]]$ which implies the vanishing of the first $2L + 1$ Fourier coefficients.

(2) $\Longrightarrow$ (1). Let us assume that (2) holds but the statement (1) is not true. By Lemma 3.1 there exists an integer l with $1 \leq l \leq L$ such that the l-shell $\Lambda_{(l)}$ is not spherical for $\mathrm{Harm}(n, d)$ for all $d = 1, \ldots, t$. As we have seen in the above remark $\Lambda_{(l)}$ is spherical for $\mathrm{Harm}(n, d)$ for all odd integers. So we may assume that $d = 2m$ is even. Now we assume that l is the minimal integer such that $\Lambda_{(l)}$ is not spherical for $\mathrm{Harm}(n, 2m)$. The minimality implies that $\Theta_{h;\Lambda}$ is a formal power series in $q^l \cdot \mathbb{R}[[q]]$ for all $h \in \mathrm{Harm}(n, 2m)$. Since the l-shell of the lattice is not spherical for $\mathrm{Harm}(n, 2m)$ we conclude that not all $\Theta_{h;\Lambda}$ are in $q^{l+1} \cdot \mathbb{R}[[q]]$. Therefore, $a_{m,2l}$ is the sum of squares of real numbers not all equal to zero and so $a_{m,2l} \neq 0$, which contradicts our assumption. □

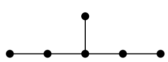
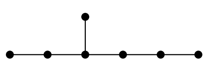
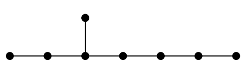
Corollary 3.4. *Let $\Lambda \subset \mathbb{E}^n$ be a lattice. If the generalized theta series $\Theta_{m,m;\Lambda}$ are zero for $m = 1, \ldots, M$, then for all integers l the l-shell of Λ is a spherical $2M + 1$ design.*

4. Root lattices — ADE

4.1. **Notation.** We recall some basic facts on root lattices. We follow here Venkov's notation from [7, Section 5]. In this section we want to consider root systems where all roots are of length one. We restrict to irreducible root systems. For a root system $R \subset \mathbb{E}^n$ we assign the following numbers: the number of roots $r = \mathrm{card}(R)$ and the Coxeter number $h = \frac{r}{n}$. For each root $\rho \in R$ we have n_0 roots σ in R which are orthogonal to ρ, and n_1 roots σ in R with $\langle \rho, \sigma \rangle = \frac{1}{2}$. These numbers n_0 and n_1 do not depend on the root ρ. They satisfy the basic equations

$$2 + n_0 + 2n_1 = r, \quad n_1 = 2h - 4, \quad n_0 = r - 4h + 6.$$

In the following table we list the irreducible root lattices with all roots of length one.

Type	Dynkin diagram	r	h	n_0	n_1
A_n		$n(n+1)$	$n+1$	$(n-1)(n-2)$	$2n-2$
D_n		$2n(n-1)$	$2n-2$	$2n^2-10n+14$	$4n-8$
E_6		72	12	30	20
E_7		126	18	60	32
E_8		240	30	126	56

4.2. **Computing the Fourier coefficient** $a_{m,2}$**.** Let $\Lambda \subset \mathbb{E}^n$ be one of the above root lattices. The next result gives us a criterion whether the 1-shell of Λ is spherical for $\mathrm{Harm}(n, 2m)$. To do so, we need to introduce some numbers. Let $x_0, x_1, x_2 \in \mathbb{E}^n$ be vectors of length one such that $\langle x_2, x_0 \rangle = 0$, and $\langle x_2, x_1 \rangle = \frac{1}{2}$ holds. We define for $a \in \{0, 1, 2\}$ the rational number $P_m(a)$ to be $p_m(x_2, x_a)$ where $p_m(x_2, x_a)$ is the bihomogeneous polynomial from 2.2. The numbers $P_m(a)$ in the following table are needed for our computation.

$P_m(a)$	$a=0$	$a=1$	$a=2$
$m=1$	$\frac{-1}{2n}$	$\frac{n-4}{8n}$	$\frac{n-1}{2n}$
$m=2$	$\frac{1}{8(n+2)(n+4)}$	$\frac{n^2-18n+8}{384(n+2)(n+4)}$	$\frac{n^2-1}{24(n+2)(n+4)}$
$m=3$	$\frac{-1}{48(n+4)(n+6)(n+8)}$	$\frac{n^3-42n^2+224n+672}{46080(n+4)(n+6)(n+8)}$	$\frac{n^3+3n^2-n-3}{720(n+4)(n+6)(n+8)}$
$m=4$	$\frac{1}{384(n+6)(n+8)(n+10)(n+12)}$	$\frac{n^4-76n^3+1148n^2+1840n-21120}{10321920(n+6)(n+8)(n+10)(n+12)}$	$\frac{n^4+8n^3+14n^2-8n-15}{40320(n+6)(n+8)(n+10)(n+12)}$

We recall for the next result the two numbers n_0 and n_1 from 4.1 which reflect the possible angles between the vectors in the 1-shell.

Proposition 4.1. *Let $\Lambda \subset \mathbb{E}^n$ be one of the above root lattices. The 1-shell of Λ is spherical for* $\mathrm{Harm}(n, 2m)$ *if and only if* $2P_m(2) + 2n_1P_m(1) + n_0P_m(0) = 0$.

Proof. We have seen in Theorem 3.3 that the 1-shell of Λ is spherical for $\mathrm{Harm}(n, 2m)$ if and only if the Fourier expansion of $\Theta_{m,m;\Lambda} = \sum_{n\geq 0} a_{m,n}q^n$ satisfies $a_{m,0} = a_{m,1} = a_{m,2} = 0$. From the Fourier expansion of $\Theta_{m,m;\Lambda}$ we can read off the Fourier coefficients

$$a_{m,n} = \sum_{(x,y)\in\Lambda\times\Lambda \text{ with } \|x\|^2+\|y\|^2=n} p_m(x, y)\,.$$

Since $p_m(x, 0) = 0 = p_m(0, y)$ the first two coefficients $a_{m,0}$ and $a_{m,1}$ are always zero. The same argument gives

$$a_{m,2} = \sum_{(x,y)\in\Lambda_{(1)}\times\Lambda_{(1)}} p_m(x, y)\,.$$

Since $p_m(x, y)$ depends only on the norms and the scalar product of the two vectors, we have for a fixed x exactly one vector $y \in \Lambda_{(1)}$ with scalar product 1, n_1 with scalar product $\frac{1}{2}$, and so on. Now since $p_m(x, -y) = p_m(x, y)$ we obtain $a_{m,2} = r(2P_m(2) + 2n_1P_m(1) + n_0P_m(0))$. □

Corollary 4.2. *The 1-shell of any root lattice of type ADE is a spherical 3-design.*

Proof. By Proposition 4.1 we have to check that the number $2P_1(2)+2n_1P_1(1)+n_0P_1(0)$ is zero. Using the formulas for n_0 and n_1 the number becomes $\frac{hn-r}{2n}$. This is zero because $r = hn$ holds — see [3, Théorème 1, V.6.2]. □

4.3. **Lattices of type A_n.** Since the 1-shell of A_1 is the same as the zero dimensional sphere S^0, it follows that this 1-shell is a spherical t-design for any t. We assume from now on that $n \geq 2$. To apply the criterion of Proposition 4.1 we compute the two numbers

$$2P_2(2) + 2n_1P_2(1) + n_0P_2(0) = \frac{(n-1)(n-2)}{96(n+2)},$$
$$2P_3(2) + 2n_1P_3(1) + n_0P_3(0) = \frac{(n-1)(n^2-16n+208)}{11520(n+8)(n+4)}.$$

Thus, we see that only the 1-shell of A_2 is spherical for $\mathrm{Harm}(n, 4)$. For all $n \geq 2$ the 1-shell of A_n is not spherical for $\mathrm{Harm}(n, 6)$.

4.4. **Lattices of type D_n.** Here we have $n \geq 4$. We compute as before

$$2P_2(2) + 2n_1P_2(1) + n_0P_2(0) = \frac{(n-4)^2}{48(n+4)},$$
$$2P_3(2) + 2n_1P_3(1) + n_0P_3(0) = \frac{(n-2)(n-16)^2}{5760(n+8)(n+4)}.$$

From the first equation we deduce that only for $n = 4$ the 1-shell of D_n is spherical for $\mathrm{Harm}(n, 4)$. We obtain that D_n is a spherical 3-design for all $n \geq 4$. Only for $n = 4$ the 1-shell of the lattice D_n is a spherical 5-design.

4.5. **The Lattices E_6, E_7, and E_8.** We compute as before:

	E_6	E_7	E_8
$2P_2(2) + 2n_1P_2(1) + n_0P_2(0)$	0	0	0
$2P_3(2) + 2n_1P_3(1) + n_0P_3(0)$	$\frac{1}{1920}$	$\frac{1}{2640}$	0
$2P_4(2) + 2n_1P_4(1) + n_0P_4(0)$	$\frac{5}{774144}$	$\frac{25}{2655744}$	$\frac{1}{71680}$

Thus, the 1-shell of all three lattices is a spherical 5-design. We see that only the 1-shell of E_8 is a 7-design. Moreover, it can be shown that the generalized theta series $\Theta_{1,1;E_6}$, $\Theta_{2,2;E_6}$, $\Theta_{1,1;E_7}$, $\Theta_{2,2;E_7}$, $\Theta_{1,1;E_8}$, $\Theta_{2,2;E_8}$, and $\Theta_{2,2;E_8}$ are zero (see [4, Section 3.4]). Therefore, the above statement is true for any l-shell of these three lattices.

Summing up, we have given a further proof of a result of B. Venkov:

Theorem 4.3 (cf. [7, Théorème 5.7])**.** *The 1-shells of the following root lattices are spherical 5-designs: A_2, D_4, E_6, E_7, and E_8. The latter gives the only 7-design.*

5. The Leech lattice Λ_{24}

The Leech lattice Λ_{24} is the unique unimodular lattice in dimension 24 with no vectors of length 1. Thus its theta series reads $\Theta_{\Lambda_{24}}(\tau) = 1 + 0 \cdot q + a_2q^2 + \ldots$. Since this series is a modular form of weight 12 for the full modular group the two known coefficients of the q-expansion of $\Theta_{\Lambda_{24}}$ determine this series uniquely. Indeed, let E_4 and E_6 be the Eisenstein series of weight 4 and 6. We have

$$E_4 = 1 + 240 \sum_{k \geq 1} \sigma_3(k)q^k \quad \text{and} \quad E_6 = 1 - 504 \sum_{k \geq 1} \sigma_5(k)q^k\,.$$

The two forms E_4^3 and E_6^2 span the vector space $M_{12}(\mathrm{SL}_2(\mathbb{Z}))$. From the first coefficients of E_4^3 and E_6^2 we see that only the linear combination $\frac{1}{12}(7E_4^3 + 5E_6^2)$ has a q-expansion starting with $1 + 0 \cdot q + \ldots$. In this way we obtain the theta series of the Leech lattice Λ_{24} as

$$\Theta_{\Lambda_{24}}(\tau) = \frac{1}{12}(7E_4^3 + 5E_6^2) = 1 + 196560q^2 + 16773120q^3 + 398034000q^4 + \ldots\,.$$

Using our criterion from Corollary 3.4 we obtain with a similar argument the next result.

Theorem 5.1 (cf. [7, Corollaire 14.3])**.** *For any integer $l \geq 2$ the l-shell of vectors λ in the Leech lattice with $\|\lambda\|^2 = l$ is a spherical 11-design.*

Proof. Since Λ_{24} is unimodular the generalized theta series $\Theta_{m,m;\Lambda_{24}}$ are modular forms in $M_{4m+24}(\mathrm{SL}_2(\mathbb{Z}))$. Considering the q-expansion of $\Theta_{m,m;\Lambda_{24}}$ given in 2.1, we see that the coefficients of q^0, q^1 , q^2, and q^3 are zero. Thus, $\Theta_{m,m;\Lambda_{24}} \in \Delta^4 \cdot M_{4m-24}(\mathrm{SL}_2(\mathbb{Z}))$. Since there are no non-zero modular forms of negative weight, we conclude that $\Theta_{m,m;\Lambda_{24}} = 0$ for $1 \leq m \leq 5$. By Corollary 3.4 this implies that the vectors of length l in the Leech lattice Λ_{24} form a spherical 11-design. □

5.1. **Other extremal lattices of small level.** [3] From Section 3 we know that for a given lattice Λ, the shells $\Lambda_{(l)}$ are t-designs for any l whenever the generalized theta series $\Theta_{m,m}(\Lambda)$ vanish for all $1 \leq m \leq \lfloor t/2 \rfloor$. Besides this fact, the proof of Theorem 5.1 contains two further ingredients. First, the ring of elliptic modular forms for the full modular group $\mathrm{SL}_2(\mathbb{Z})$ is generated by an Eisenstein series E_4 and a cuspidal modular form Δ, which has a simple zero at the only cusp at infinity. Secondly, the observation that $\Theta_{m,m}(\Lambda_{24})$ vanishes to order 4 at the cusp.

Therefore – this being referee's observation –, the proof above goes through for any (even) extremal unimodular lattice in dimension n, for any n divisible by 24. Indeed, in this case *extremal* means that the lattice first minimum is at least $1 + \lfloor (n + 4m)/12 \rfloor$. Hence, since the weight of $\Theta_{m,m}(\Lambda)$ is $n + 4m$, we have that $n + 4m - 12(2 + n/12)$ has to be non-negative, i.e. $m \geq 6$ as in the proof of our last theorem. The same argument gives the other two results for the unimodular case from [1, Cor. 4.1] – cf. loc.cit. for the complete statement.

In the modular, non unimodular case, the same strategy as before works when the level N of the lattice Λ satisfies $N + 1 \mid 24$. In this case, the ring of modular forms with respect to the Fricke group $\Gamma_*(N)$ is again generated by an Eisenstein series and a cusp form Δ_N with simple zero at the single cusp – see [6] for undefined notation used in this subsection and further details. The following proposition shows that for modular lattices the generalized theta series are indeed modular forms for the Fricke group.

Proposition 5.2. *Let $\Lambda \subset \mathbb{E}^n$ be an integral, N-modular lattice, $n \equiv 0 \pmod 4$ and m a positive integer. Then for any $\gamma \in \Gamma_*(N)$ we have*

$$\Theta_{m,m}(\Lambda) \mid_k \gamma = \Theta_{m,m}(\Lambda),$$

where $k = n + 4m$ is the weight.

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[3] This subsection stems from the referee's encouragement to reprove [1, Cor. 4.1], at least for the unimodular cases. We thank her/him for this suggestion.

Proof. It follows from a straightforward calculation using the $\mathrm{O}(n)$-invariance of $\Theta_{m,m}$ from Theorem 2.1. Namely,

$$(\Theta_{m,m}(\Lambda)\mid_k \gamma)(z) := j(\gamma,z)^{-k}\,\Theta_{m,m}(\gamma z) = j(\gamma,z)^{-k}\sum_{h\in\mathcal{B}_{\mathrm{harm}}}\Theta_{h,\Lambda}(\gamma z)^2 =$$
$$\sum_{h\in\mathcal{B}_{\mathrm{harm}}}\left(j(\gamma,z)^{-k/2}\Theta_{h,\Lambda}(\gamma z)\right)^2 = \sum_{h\in\mathcal{B}_{\mathrm{harm}}}\left(\Theta_{h,\Lambda}\mid_{k/2}\gamma\right)^2(z) =$$
$$\sum_{h\in\mathcal{B}_{\mathrm{harm}}}\left(\chi_{n/2}(\gamma)\Theta_{h,\Lambda'}(z)\right)^2 = \sum_{h\in\mathcal{B}_{\mathrm{harm}}}\Theta_{h,\Lambda'}(z)^2 = \Theta_{m,m}(\Lambda')(z) = \Theta_{m,m}(\Lambda)(z),$$

where from the second to the third line we use a standard calculation for the action of the Fricke involution on theta series with harmonic coefficients (cf. [1, Cor. 3.1]), and the last equality follows from the $\mathrm{O}(n)$-invariance property of the generalized theta series – since Λ is assumed to be N-modular and so $\Lambda \cong \Lambda' := \sqrt{N}\Lambda^{\#}$. □

Remark 5.3. In the situation of the proposition above, the theta series with harmonic coefficients $\Theta_{h,\Lambda}$ are not in general *Fricke modular forms*, i.e. modular forms for the Fricke group. Instead, the sum and the difference of $\Theta_{h,\Lambda}$ and $\Theta_{h,\Lambda'}$ are – for different characters though! This fact is used in [1] together with a vanishing condition for these two Fricke modular forms (Proposition 4.1 loc.cit.) to characterize further spherical designs coming from shells of N-modular lattices.

We exemplify the aforementioned strategy on a 2-modular case.

Corollary 5.4 (cf. [1, Cor. 4.1, from Case $l = 2$]). *Let $\Lambda \subset \mathbb{E}^n$ be an even, extremal, 2-modular lattice and $n \equiv 4 \pmod{16}$. Then all the shells of Λ are spherical 5-designs.*

Proof. Writing $n = 16n' + 4$ we have that the minimum of Λ is at least $1 + \lfloor (n+4m)/8 \rfloor = 1 + 2n'$, hence at least $2 + 2n'$ since the lattice is even. From Proposition 5.2 we have that $\Theta_{m,m}(\Lambda)$ is a modular form for the Fricke group $\Gamma_*(2)$. The ring structure of the Fricke modular forms described in [6] allows us to write $\Theta_{m,m}(\Lambda)$ as a Fricke modular form times $\Delta_2^{2+2n'}$ ($\Delta_{16}^{2+2n'}$ in the notation of [6]) and therefore $n + 4m - 8(2+2n')$ must be non-negative whenever $\Theta_{m,m}(\Lambda)$ is non-zero. Consequently, after Theorem 3.3 all shells of Λ are spherical 5-designs, since $\Theta_{m,m}(\Lambda)$ vanishes identically for $m = 1, 2$. □

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