Shimura varieties and canonical models

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1 Introduction

Consider the following three examples of connected complex manifolds: the upper (hyperbolic) half-plane $\mathcal{H}_1$, the affine space $\mathbb{A}^1(\mathbb{C})$ and the projective space $\mathbb{P}^1(\mathbb{C})$. Each can be given a hermitian metric such that the respective actions of $\mathrm{PSL}_2(\mathbb{R})$, $\mathbb{G}_a(\mathbb{C})$ and $\mathrm{PU}_2(\mathbb{C})$ are isometric and transitive. We will restrict attention to spaces like $\mathcal{H}_1$ of negative curvature (i.e. 'diverging geodesics') called **hermitian symmetric domains (hsd)**. If the (Lie-)group of isometries, Iso(D), of such a space D acts transitively we may ask for an algebraic group G_D over $\mathbb{R}$ whose real points realize Iso(D). There are precise conditions for a group to have real points of this kind, placing us in the realm of algebraic groups. Choosing a point in the space allows to get D back from G.

Taking this axiomatic point of view we define arithmetic subgroups of $\Gamma \subseteq G_D(\mathbb{R})$ and ask when the quotient is a) compact and b) an algebraic variety. The answer to both these questions is given by the **Baily-Borel** theorem: If Γ is torsion-free the quotient space $D(\Gamma) := \Gamma\backslash D$ is an open subvariety in a projective variety $D(\Gamma)^*$.

To make full use of this viewpoint, we introduce the circle group $U_1 := \{z \in \mathbb{C}^\times \mathrm{with} |z| = 1\}$ and show that the points of a hsd. D correspond to morphisms $u_p \colon U_1 \to G_D$ that operate on the Lie algebra via the characters $z, 1, z^{-1}$ (and some other conditions). To avoid choosing a base point in D, we consider instead of a single morphism $U_1 \to G$ an entire $G(\mathbb{R})^+$-conjugacy class X^+ of morphisms. This pair (G, X^+) is called a **connected Shimura datum**. The connected Shimura variety $\mathrm{Sh}^\circ(G, X^+)$ is the projective system over the spaces $\Gamma\backslash D$, where Γ runs over the torsion-free arithmetic subgroups or, equivalently, it is the adélic quotient $G(\mathbb{Q})\backslash(D \times G(\mathbb{A}_f))$.

Coming back to the example of $\mathcal{H}_1$ we know that its quotients by arithmetic groups are moduli spaces: they parametrize elliptic curves with level structure. What do the quotients $\Gamma\backslash D$ of a Shimura datum parametrize?

To answer this, note that there is another setting in which morphisms from the circle group come up: **Hodge structures**. A Hodge structure on a real vector space V is a $\mathbb{Z} \times \mathbb{Z}$-grading on its complexification compatible with conjugation. Equivalently, a Hodge structure can be given by (the characters of) a representation of the real torus $\mathbb{S} := \mathbb{G}_{m/\mathbb{C}}$ on V. Now note that the group U_1 from above is the norm 1 subgroup of $\mathbb{S}$. Let $V_\mathbb{C}$ be the tangent space of the manifold $G_\mathbb{C}^{\mathrm{ad}}$ obtained from a connected Shimura datum. A hermitian metric on this space is a smoothly varying pairing on $V_\mathbb{C}$, called a polarization. One can show that every hermitian symmetric domain is a moduli space for polarized Hodge structures on some vector space.

For arithmetic purposes it is too restrictive to consider only connected domains and the identity component $G(\mathbb{R})^+$ of a semisimple algebraic group. Hence, general Shimura data (G, X) are built from certain reductive groups G and $G(\mathbb{R})$-conjugacy classes X of morphisms $\mathbb{S} \to G_\mathbb{R}$.
Let S be a scheme over $\mathbb{C}$, then a **model** for S over a number field k is a scheme S_0 over k with an isomorphism $S_0 \otimes_k \mathbb{C} \cong S$. In general we can expect neither existence nor uniqueness of models. For example, an elliptic curve E over $\mathbb{C}$ has a model over some number field if and only if $j(E)$ is algebraic. The main insight of Shimura is the existence of models of Shimura varieties over number fields. In the later talks we will discuss existence of canonical models for several families of Shimura varieties.

2 Talks

The first two talks aim to give a summary of (affine) algebraic groups over a perfect field k with a bias towards examples.

Talk 1 (Algebraic groups I)**.**
Following [3], define *algebraic groups* over a field k and explain the statement of the theorem of Barsotti-Chevalley.
Leaving abelian varieties aside, we focus in the following on affine group schemes (**ags**). Give some examples of these: $\mathbb{G}_m, \mathbb{G}_a, \boldsymbol{\mu}_n, \mathrm{GL}_n, \mathrm{Sp}_n, \mathrm{U}_n$ – not necessary all of these at this stage. Give ways of constructing further **ags**: diagonal **ags**, constant **ags** – the speaker may skip these if time requires. Define representations, and very briefly deduce that linear algebraic groups are **ags**. Discuss connected components, étale **ags**, the short exact sequence $1 \to G^0 \to G \to \pi_0(G) \to 1$.
Define **ags** of multiplicative type, character group, tori. Mention the field of definition of tori, the equivalence of categories between tori and free Galois-modules. Define K/k-forms of **ags** and give the correspondence with one cocycles. As examples for K/k-forms give at least (non-split) tori and the Weyl restriction.
(*)Develop the necessary concepts to define semi-simple and reductive groups – and show their structure up to isogeny.

References: First one needs [3] for the statement of the theorem of Barsotti-Chevalley. Then the speaker is encouraged to follow [19], Chapter I, §§1–4.2. Certainly one can/should also look at [21], [14] or [20].
Difficulty: ** – comprehensive but self-contained
Date: April 16, 2014 **Speaker: N.N.**

Talk 2 (Algebraic groups II)**.** Finish whatever remains to be explained from last talk's (*) – if indeed necessary, we could then start at 9:00 so that the first speaker finishes with her/his definitions and at 9:15 the second speaker continues.
Explain Borel's result on fixed points (a connected solvable **ags** acting on a complete algebraic variety has a fixed point) – which is Lie-Kolchin for $G = \mathrm{GL}_n$. Up to here one has gone through [19], Chapter I, §§1– 4.3.
For the rest of the talk, we do some Lie theory for algebraic groups and then indicate the classification of semi-simple **ags** at least for $k = \mathbb{R}$.
For the Lie theory we may follow [14]. Define the Lie algebra of an **ags**, give examples, mention the property of the functor Lie when $\mathrm{char}(k) = 0$ for connected **ags**. Define the adjoint map Ad and the group G^{ad}. Illustrate that when G is connected, then G^{ad} is precisely (isomorphic to!) the image of Ad. If G is semi-simple, then G^{ad} is a direct product of simple groups. Finish this discussion by mentioning the structure of reductive groups ([14, Theorem 15.1]).
The remaining time should be used to explain the classification of semi-simple **ags** (as done in [19]) via the Γ-diagrams for G (over the reals!). For this, one may reduce the task to simply explain the definition of a Γ-diagram and then show the corresponding classifying list given in the reference. One may also list the groups from [20, Chapter 17].
References: For the first part of the talk, as well as the last one, one needs [19] (and also the last chapter of [20] for the list). For the Lie theory we follow [14].
Difficulty: ** – same as talk 1, few proofs but a lot of material to understand
Date: April 23, 2014 **Speaker: N.N.**

Talk 3 (Hermitian symmetric domains)**.**
All references are to Milne, [15]. This talk gathers necessary results from riemannian/hermitian geometry. The emphasis here should lie on examples rather than proofs. The must-do proof is Thm. 1.21. Follow §1 in [15] and explain/define almost-complex structures, hermitian manifolds and hermitian symmetric spaces, i.e. where $\mathrm{Iso}(M, g)$ acts transitively and every point is isolated fixpoint of an involution. Discuss the examples $\mathcal{H}_1, \mathbb{C}/\Lambda$ and $\mathbb{P}^1(\mathbb{C})$ from above and how they fit in the general classification. Explain the Bergmann metric: state Thm. 1.3 without proof, but give the remarks after it.

Then consider the automorphism groups of these spaces and discuss the relation to Lie groups and algebraic groups: 1.5–1.7. Explain the correspondence between points in a hsd and certain representations of U_1 (Thm. 1.9, with sketch of proof, if time permits). Discuss in detail: Cartan involutions with examples 1.15 and 1.17. Define C-polarization and give the connection to Cartan involution by Thm. 1.16 and Prop. 1.20. Finally, prove Thm. 1.21. and deduce Cor. 1.22.
Further **reference**: For an overview and examples, cf. [8], part 1, pp. 11–18.
Difficulty: * – not so difficult if you know your differential geometry
Date: April 30, 2014 **Speaker: N.N.**

Talk 4 (Locally symmetric varieties and their compactifications)**.**
Follow §3 in [15]: Define arithmetic and congruence subgroups (p.42), locally symmetric varieties and discuss quotients by torsion free discrete subgroups (Prop. 3.1 and Thm. 3.3), give Expl. 3.4 (quaternion groups), state the important Thm. 3.12 (Baily-Borel), explain the proof for $\mathcal{H}_1$ (following Milne), cf. [22] for one more example (Hilbert modular surfaces). Next, give Thm. 3.14, Cor. 3.16 and Thm. 3.21.
Further reference: Milne only gives a historical overview of Thm. 3.12. The speaker is encouraged to look at/explain some parts of [18] for more background.
Difficulty: ** – a lot of background material; definitely on the difficult side
Date: May 7, 2014 **Speaker: N.N.**

Talk 5 (Variations of Hodge structures)**.**
The aim of this talk is to explain §1.1 of Deligne, [5]. The material is also covered by §2 of Milne, [15]. The following should be explained: Hodge structures (HS) and filtrations, pure HS, rational/integral HS, examples [15] 2.4-2-6: complex structure $\leftrightarrow (-1,0)(0,-1)$ type HS, Tate HS $\mathbb{Q}(m)$; examples coming from geometry, in particular abelian varieties, cf. §1-3 of [17]; representations of the Deligne-Torus $\mathbb{S}$, weight homomorphism w_h and μ_h, category of HS, polarizations of HS, continuous/holomorphic family of HS, variation of HS (cover flag varieties only as much as necessary). State and prove Thm. 1.1.14 in [5] relating hsd with a variation of HS; cf. Thm. 2.14 in [15].
Further reference: The fact that $\mathbb{S}$-representations come up in this context is part of the general Tannakian picture: cf. Remarks 2.30,2.31 in [6].
Difficulty: * – self-contained and concrete, but many topics
Date: May 14, 2014 **Speaker: N.N.**

Talk 6 (Adéles and connected Shimura varieties)**.**
All references are to [15].
Recall the finite adéles of $\mathbb{Q}$ and the topology on adélic points $G(\mathbb{A}_f)$ of an algebraic group (Prop. 4.1). Explain equivalence of Def. 4.4 and Def. 4.22 (axioms SV1-SV3) for a connected Shimura datum. Sketch the proof of Prop. 4.8 using the theorem from talk 3. Next, define connected Shimura varieties as in 4.10-4.12. The examples in 4.14 should be explained in detail. Give the strong approximation theorem, maybe some examples for it, and use it to prove an adélic description of connected Shimura varieties, Prop. 4.18. At least state Prop. 4.19.
Difficulty: * – not difficult, but needs previous talks 3-5
Date: May 21, 2014 **Speaker: N.N.**

Talk 7 (General Shimura varieties and Tori)**.**
This talk defines general Shimura varieties and their morphisms. It then gives an important structural result: If G^{der} is simply connected, $\pi_0(\mathrm{Sh}(G,X))$ is a 0-dimensional Shimura variety. References are to Milne [15].
First give some results on the real points of an algebraic group over $\mathbb{R}$: In particular state the theorem of Cartan and real approximation (5.1–5.4). Then give Def. 5.5 of a Shimura datum and check the assertions for the standard example GL_2 (Exple. 5.6). Note that 5.9 follows from 1.1.14 of [5], which

was proven in talk 5 . Prove 5.13 and finally define the Shimura variety attached to a Shimura datum. Define morphisms of Shimura varieties and give Thm. 5.16 (without proof).

State and prove Thm. 5.17: The maximal abelian quotient $G \to T$ induces an isomorphism of $\pi_0(\mathrm{Sh}_K(G,X))$ to a Shimura variety for T. This leads us to introduce 0-dimensional Shimura varieties. State the additional axioms SV4-SV6 and discuss Expl. 5.24.
Difficulty: ** – needs good understanding of previous talks 4 and 6
Date: May 28, 2014 **Speaker: N.N.**

Talk 8 (Siegel modular varieties)**.**
In the first part of this talk, we describe the complex points of the Shimura varieties associated to $G = \mathrm{GSp}(V)$, for V a symplectic $\mathbb{Q}$-vector space. The slides of U. Görtz ([8, §3 Talk 2]) may serve as orientation, and the speaker is encouraged to fill out the necessary details from [10, Chapter 1] and/or from [7, Chapter 1].
The second part of the talk is more ambitious and aims to roughly explain the construction of the moduli space of polarized abelian schemes. For this one may summarize the content of [7, Chapter 2] – where we certainly use GIT as a black box, in particular the construction of the Hilbert scheme. The stress in this second part should be put to make plausible the equivalence in Proposition 2.6.1 and the description of $\mathcal{A}_{n,N}$ (both on page 19 loc.cit.).
References: For the first part we need [8, §3 Talk 2], [10, Chapter 1] and/or [7, Chapter 1]; and [7, Chapter 2] for the second one.
Difficulty: *** – Background reading of several sources required, difficult material
Date: June 4, 2014 **Speaker: N.N.**

Talk 9 (Abelian varieties of CM-type)**.**
This talk is independent of all the others. All references are to [15], from which §10–§11 should be presented.
From §10: Short reminder on abelian varieties (AV), CM-fields and CM-types, every CM-type AV has an algebraic model, Prop. 10.3 in [15], good reduction of AV, Main theorem of CM-theory: Shimura-Taniyama formula. From §11: Convention for the Artin map, reflex field of a CM-type; for further details cf. [16]
Difficulty: ** – independent, but background in CM-theory or considerable reading required
Date: June 11, 2014 **Speaker: N.N.**

Talk 10 (Canonical models of Shimura varieties)**.**
In this talk we define the notion of a canonical model for a Shimura variety. We proof its uniqueness by the following argument[1] A variety over a field k is determined by the Galois action on the base change to k^{ac}. We then show, that there exists a dense set of 'special' points whose combined Galois action exhausts $G(k^{\mathrm{ac}}/k)$ and hence uniquely determine a model over k.
First recall the central statements of class field theory and fix a convention for the Artin map, §11 in [15], 0.8 in [5]. From §12 in [15] cover: models, reflex fields of Shimura data, Remark 12.3b, examples 12.4 (c-d) and 12.7, special points, canonical model of $\mathrm{Sh}_K(G,X)$ and $\mathrm{Sh}(G,X)$.
Show uniqueness as in §13 of [15], see 5.1-5.5 of [5] for the proof of the key lemma.
Difficulty: **(*) – needs CM-theory plus a working understanding of previous talks 6–7
Date: June 18, 2014 **Speaker: N.N.**

Talk 11 (Canonical models for Siegel and elliptic modular varieties)**.**
This is possibly a shorter talk presenting the first full examples of the theory.
For canonical models of elliptic modular curves cf. [13], pp. 1–32. This should pose no problems and need not be overly long. The more difficult Siegel case is in pp. 110–115 of [15]. One should in particular prove Prop. 14.10 (using Mumford, where 'Hodge group' refers to the Mumford-Tate

[1] See also 'Where we are headed' in §10 of [15].

group of today's terminology). The fundamental theorem for CM-algebras (as opposed to fields) can be found in [16]
Difficulty: ** – moderately difficult, detailed sources
Date: June 25, 2014 **Speaker: N.N.**

Talk 12 (Example: Picard surfaces I).
This and the next talk introduce arithmetic quotients of the open unit-ball in $\mathbb{C}^2$, so called Picard modular surfaces. All references are to Gordon [9].
Following §1, recall the necessary notations of signature and maximal compact subgroup of unitary groups, then give the various descriptions of the symmetric domain X for $G' \cong \mathrm{SU}(2,1)$, and sketch the Baily-Borel and smooth compactifications of $S_\Gamma(\mathbb{C}) := \Gamma\backslash X$. In §2, introduce the similitude norm μ and the reductive group $G_{/\mathbb{Q}}$, such that $G' = \ker(\det/\mu)$. Explain the splitting 2.1.1 and 2.1.2. Describe the Shimura datum (G,X), the complex structure on X and the connected components of the Shimura variety $S_K(G,X)(\mathbb{C})$, Lemma 2.4. In the third part, we give a moduli interpretation of its points: Define the signature of a polarized CM abelian variety A and level-K structures on A. Proof the main result of this talk: Prop. 3.2.
Difficulty: ** – self-contained but not very detailed source
Date: July 2, 2014 **Speaker: N.N.**

Talk 13 (Example: Picard surfaces II).
Following §4 of Gordon [9], recall the reflex field $E(G,X)$ of $h \in X$ and show Lemma 4.2, i.e. $E(G,X) = E$. Definitions 4.4 and 4.5 already appeared in talk 10, so just give a short reminder and then proof Prop. 4.6, the existence of a model over E. Give the Shimura reciprocity law r and proof Thm 4.9.
From §5 one can give some remarks on the compactifications and their moduli interpretation, but anything more would be a considerable effort, cf. Larssen [12].
In the remaining time sketch the generalization from an imaginary quadratic to a CM field E, including the moduli interpretation of $\mathrm{Sh}_K(G,X)(\mathbb{C})$ by weak polarizations, Prop. 6.3.2. Determine the reflex field (6.4.1.) and proceed as in Thm 4.9 or [5],5.7, to show existence of a canonical model.
Difficulty: ** – same as previous talk
Date: July 9, 2014 **Speaker: N.N.**

Talk 14 (Possible Example: Shimura curves).
This would be an ambitious endpoint to the seminar. Whether we will have time enough (and people willing to prepare a talk) remains open for now.
4 sources: Milne's article [13] introduces only the modular curve case. Carayol's (difficult) article [1] is the ultimate reference. To get starting with this matter, maybe Jarvis [11] can be of help. Finally, Buzzard has an article, where he deals with the (easier) exceptions to Carayol's Theorem.
Difficulty: *** – difficult
Date: July 16, 2014 **Speaker: N.N.**

Talk 15 (Continuation of previous talk or Make-up day).
Date: July 23, 2014 **Speaker: N.N.**

References

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